

Supporting Information

SI MATLAB codes for generating Figures in the manuscript.

Fig. 1

```
clc
clear all
b=1;y=linspace(-b,b,100);alphaS=2;
ALL_f=[0.1 0.5 1 1.9];
ALL_yi=zeros(2*length(ALL_f),2*length(y));ALL_zi=ALL_yi;
for i=1:length(ALL_f)
    f=ALL_f(i);
    z2=(-b^2/2*f+b^2-y.^2)/alphaS^2;
    indi=find(z2>0);
    zi=[-(z2(indi)).^0.5 (z2(indi)).^0.5];
    ai(i)=b/alphaS*(1-f/2)^0.5;bi(i)=b*(1-f/2)^0.5;Aci(i)=pi*b^2/alphaS*(1-f/2);
    ALL_zi(i,1:length(zi))=zi;
    ALL_yi(i,1:length(zi))=[y(indi) y(indi)];
end
Ac=pi*ai.*bi
Aci
test=ALL_zi(1,:);ind1=find(test~=0);z1=ALL_zi(1,ind1);y1=ALL_yi(1,ind1);
test=ALL_zi(2,:);ind2=find(test~=0);z2=ALL_zi(2,ind2);y2=ALL_yi(2,ind2);
test=ALL_zi(3,:);ind3=find(test~=0);z3=ALL_zi(3,ind3);y3=ALL_yi(3,ind3);
test=ALL_zi(4,:);ind4=find(test~=0);z4=ALL_zi(4,ind4);y4=ALL_yi(4,ind4);

Theta=linspace(0.5,2,100);
E=0.5./Theta.^3;

FontSize=17;
subplot(1,2,1)
h1=plot(z1,y1,'sk',z2,y2,'o',z3,y3,'*',z4,y4,'v')
title('(a)', 'FontSize',FontSize)
legend(['u* =' num2str(ALL_f(1))],[ 'u* =' num2str(ALL_f(2))],...
    ['u* =' num2str(ALL_f(3))],[ 'u* =' num2str(ALL_f(4))], 'Location','Best')
xlabel('z/b', 'FontSize',FontSize);ylabel('y/b', 'FontSize',FontSize)
set(h1, 'LineWidth',2);
set(gca, 'FontSize',FontSize);
subplot(1,2,2)
h2=plot(Theta,E)
title('(b)', 'FontSize',FontSize)
xlabel('\Theta', 'FontSize',FontSize);ylabel('E(\Theta)', 'FontSize',FontSize)
set(h2, 'LineWidth',2);
set(gca, 'FontSize',FontSize);
```

Fig. 2

```
clear all
clc
DataPoints=1e3;
yPb=linspace(-2/3,4/3,DataPoints);
fmax=20/9;
ALL_f=[0.1 1 0.9*fmax];
for i=1:length(ALL_f)
    f=ALL_f(i);
    test=(32/27+yPb.^3-2*yPb.^2-8/15*f)./(3*yPb+2);
    ind=find(test<0);test(ind)=0;
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```

        zPb2(i,:)=test;
end
x1=linspace(-2/3^0.5,0,DataPoints);y1=3^0.5*x1+4/3;
x2=linspace(0,2/3^0.5,DataPoints);y2=-3^0.5*x2+4/3;
x3=linspace(-2/3^0.5,2/3^0.5,DataPoints);y3=-2/3*ones(1,length(x3));

mz1=zPb2(1,:).^0.5;ind_mz1=find(mz1>0);mz11=-mz1(ind_mz1);my11=yPb(ind_mz1);
pz1=zPb2(1,:).^0.5;ind_pz1=find(pz1>0);pz11=pz1(ind_pz1);py11=yPb(ind_pz1);

mz2=zPb2(2,:).^0.5;ind_mz2=find(mz2>0);mz22=-mz2(ind_mz2);my22=yPb(ind_mz2);
pz2=zPb2(2,:).^0.5;ind_pz2=find(pz2>0);pz22=pz2(ind_pz2);py22=yPb(ind_pz2);

mz3=zPb2(3,:).^0.5;ind_mz3=find(mz3>0);mz33=-mz3(ind_mz3);my33=yPb(ind_mz3);
pz3=zPb2(3,:).^0.5;ind_pz3=find(pz3>0);pz33=pz3(ind_pz3);py33=yPb(ind_pz3);

FontSize=28;
h=plot(mz11,my11,'ob',mz22,my22,'vr',mz33,my33,'*g',x1,y1,'sk',x2,y2,'sk',...
        x3,y3,'sk',pz11,py11,'ob',pz22,py22,'vr',pz33,py33,'*g')
xlabel('z/b','FontSize',FontSize);ylabel('y/b','FontSize',FontSize)
legend(['u* = ' num2str(ALL_f(1))],[ 'u* = ' num2str(ALL_f(2))],...
        ['u* = ' num2str(ALL_f(3))], 'Location','NorthEast')
set(h,'LineWidth',2);
set(gca,'FontSize',FontSize);

```

Fig. 3

```

clear all
clc
DataPoints=1e3;

ymin=-2/3;ymax=4/3;

yPb=linspace(ymin,ymax,DataPoints);
ind_my=find(yPb<0);ind_py=find(yPb>0);
fmax=20/9;
AtPb2=4/3^0.5;
ALL_f=linspace(1e-3,0.99*fmax,DataPoints);
for i=1:length(ALL_f)
    f=ALL_f(i);
    test=(32/27+yPb.^3-2*yPb.^2-8/15*f)./(3*yPb+2);
    ind=find(test<0);test(ind)=0;
    zPb(i,:)=test.^0.5;
end

Delta_y=diff(yPb);
for j=1:length(ALL_f)
    zj=zPb(j,:);
    half_Ac(j)=sum(abs(Delta_y.*zj(1:end-1)));
end
Ac=2*half_Ac;

FontSize=28;
h=plot(ALL_f,Ac/AtPb2)
xlabel('u*','FontSize',FontSize);ylabel('A_c/A_{ct}','FontSize',FontSize)

```

```
set(h,'LineWidth',2);
set(gca,'FontSize',FontSize);
```

Fig. 4

```
clear all
clc
DataPoints=1e3;

ymin=-2/3;ymax=4/3;

yPb=linspace(ymin,ymax,DataPoints);
ind_my=find(yPb<0);ind_py=find(yPb>0);
fmax=20/9;
AtPb2=4/3^0.5;
ALL_f=linspace(1e-3,0.99*fmax,DataPoints);
for i=1:length(ALL_f)
    f=ALL_f(i);
    test=(32/27+yPb.^3-2*yPb.^2-8/15*f)./(3*yPb+2);
    ind=find(test<0);test(ind)=0;
    zPb(i,:)=test.^0.5;
end
Delta_y=diff(yPb);
for j=1:length(ALL_f)
    zj=zPb(j,:);
    half_Ac(j)=sum(abs(Delta_y.*zj(1:end-1)));
end
Ac=2*half_Ac;AcPAct=Ac/AtPb2;
dAcPAct=diff(AcPAct);df=diff(ALL_f);
ALL_G=abs(dAcPAct./df);
ALL_Theta=1./ALL_f(1:end-1);Theta_min=1/fmax;
ind_theta=find(ALL_Theta>Theta_min);Theta=ALL_Theta(ind_theta);
G=ALL_G(ind_theta);
E=Theta.^(-3).*G;
[SorttedTheta ind_Sort]=sort(Theta);Sorted_E=E(ind_Sort);
INT_E=trapz(SorttedTheta,Sorted_E)
n=(2-Theta_min)/(1-Theta_min);n=round(n*100)/100;
K=(n-1)*Theta_min^(n-2);K=round(K*100)/100;
E_cal=K*Theta_min./Theta.^n;
x0=E;y0=E_cal;
R0_Pearson=sum((x0-mean(x0)).*(y0-mean(y0)))/(sum((x0-mean(x0)).^2)*sum((y0-mean(y0)).^2))^0.5;
R2=round(R0_Pearson^2*100)/100

FontSize=12;
h=loglog(Theta,E,'s',Theta,E_cal)
xlabel('\theta','FontSize',FontSize);ylabel('E(\theta)','FontSize',FontSize)
legend('Numerical values',[ 'E = ' num2str(K) '\theta_{min}/\theta^{ ' num2str(n) ' }'], 'Location','NorthEast')
set(h,'LineWidth',2);
set(gca,'FontSize',FontSize);
```

Fig. 5

```
clear all
clc
DataPoints=1e3;
```

```

ymin=-2/3;ymax=4/3;

yPb=linspace(ymin,ymax,DataPoints);
ind_my=find(yPb<0);ind_py=find(yPb>0);
fmax=20/9;
AtPb2=4/3^0.5;
ALL_f=linspace(1e-3,0.99*fmax,DataPoints);
for i=1:length(ALL_f)
    f=ALL_f(i);
    test=(32/27+yPb.^3-2*yPb.^2-8/15*f)./(3*yPb+2);
    ind=find(test<0);test(ind)=0;
    zPb(i,:)=test.^0.5;
end
Delta_y=diff(yPb);
for j=1:length(ALL_f)
    zj=zPb(j,:);
    half_Ac(j)=sum(abs(Delta_y.*zj(1:end-1)));
end
Ac=2*half_Ac;AcPAct=Ac/AtPb2;
dAcPAct=diff(AcPAct);df=diff(ALL_f);
ALL_G=abs(dAcPAct./df);
ALL_Theta=1./ALL_f(1:end-1);Theta_min=1/fmax;
ind_theta=find(ALL_Theta>Theta_min);Theta=ALL_Theta(ind_theta);
G=ALL_G(ind_theta);
E=Theta.^(-3).*G;
[SorttedTheta ind_Sort]=sort(Theta);Sorted_E=E(ind_Sort);
INT_E=trapz(SorrtdTheta,Sorted_E)
n=(2-Theta_min)/(1-Theta_min);n=round(n*100)/100;
K=(n-1)*Theta_min^(n-2);K=round(K*100)/100;
E_cal=K*Theta_min./Theta.^n;
x0=E;y0=E_cal;
R0_Pearson=sum((x0-mean(x0)).*(y0-mean(y0)))/(sum((x0-mean(x0)).^2)*sum((y0-mean(y0)).^2))^0.5;
R2=round(R0_Pearson^2*100)/100

FontSize=12;
h=loglog(Theta,E,'s',Theta,E_cal)
xlabel('\theta','FontSize',FontSize);ylabel('E(\theta)','FontSize',FontSize)
legend('Numerical values',[ 'E = ' num2str(K) '\theta_{min}/\theta^{ ' num2str(n) ' }'], 'Location','NorthEast')
set(h,'LineWidth',2);
set(gca,'FontSize',FontSize);

```

Fig. 6

```

clc
clear all
DataPoints=1e3;
b=1;
ALL_phi=[pi/6 pi/4 pi/3];
for i_phi=1:length(ALL_phi)
    phi=ALL_phi(i_phi);
    a=b/(2*cos(phi)),alphaS=b/a
    invM=((alphaS^4/2+2*alphaS^2-1)*phi-8/3*alphaS^3*sin(phi)+(alphaS^2- ...
2/3)*sin(2*phi)-sin(4*phi)/12)/((2-alphaS^2)*phi+sin(2*phi))
    M=invM^(-1);
    fmax=M*(1+alphaS-3/4*alphaS^2)*(alphaS/2-1)/(alphaS/2+1)

```

```

Theta=linspace(0,phi,DataPoints);r_out=2*a*cos(Theta);
fj=[0.5 1 2];
dr=zeros(length(fj),DataPoints-1);
for j=1:length(fj)
    for i=1:DataPoints
        rPa=linspace(alphaS,r_out(i)/a,DataPoints);
        f_Theta=M*(rPa.^2-alphaS.^2).*(1-2*cos(Theta(i))./rPa);
        [error ind_i]=min((1-f_Theta/fj(j)).^2);
        if error<1e-4
            rPa_j(j,i)=rPa(ind_i);Theta_j(j,i)=Theta(i);
        else
            rPa_j(j,i)=0;Theta_j(j,i)=0;
        end
    end
end

ALL_r_L=zeros(length(fj),DataPoints-1);ALL_r_R=ALL_r_L;
ThetaL=ALL_r_L;ThetaR=ALL_r_L;drL=ALL_r_L;drR=ALL_r_L;
dif1=(rPa_j(1,1:DataPoints-1)-alphaS)*a;
dif2=r_out(1:DataPoints-1)-rPa_j(1,1:DataPoints-1)*a;
[drj,ind_drj]=min([dif1;dif2],[],1);
indL_drj=find(ind_drj==1);r_L=rPa_j(1,indL_drj)*a;
indR_drj=find(ind_drj==2);r_R=rPa_j(1,indR_drj)*a;
ALL_r_L(1,1:length(indL_drj))=r_L;ThetaL(1,1:length(indL_drj))=Theta(indL_drj);
ALL_r_R(j,1:length(indR_drj))=r_R;ThetaR(j,1:length(indR_drj))=Theta(indR_drj);

for j=1:length(fj)
    Delta_r=rPa_j(j,1:DataPoints-1)*a-b;[MAXTheta MAXind]=max(Theta_j(j,:));
    indL_drj=find(Delta_r<Delta_r(MAXind));r_L=rPa_j(j,indL_drj)*a;
    indR_drj=find(Delta_r>Delta_r(MAXind));r_R=rPa_j(j,indR_drj)*a;

    ALL_r_L(j,1:length(indL_drj))=r_L;ThetaL(j,1:length(indL_drj))=Theta(indL_drj);

    ALL_r_R(j,1:length(indR_drj))=r_R;ThetaR(j,1:length(indR_drj))=Theta(indR_drj);
end

X0(i_phi,:)=[b.*cos(Theta) b.*cos(Theta)]/a;
Y0(i_phi,:)=[b.*sin(Theta) -b.*sin(Theta)]/a;
Xf(i_phi,:)=[r_out.*cos(Theta) r_out.*cos(Theta)]/a;
Yf(i_phi,:)=[r_out.*sin(Theta) -r_out.*sin(Theta)]/a;

X1L(i_phi,:)=[ALL_r_L(1,:).*cos(ThetaL(1,:)) ALL_r_L(1,:).*cos(ThetaL(1,:))]/a;
Y1L(i_phi,:)=[ALL_r_L(1,:).*sin(ThetaL(1,:)) -ALL_r_L(1,:).*sin(ThetaL(1,:))]/a;
X1R(i_phi,:)=[ALL_r_R(1,:).*cos(ThetaR(1,:)) ALL_r_R(1,:).*cos(ThetaR(1,:))]/a;
Y1R(i_phi,:)=[ALL_r_R(1,:).*sin(ThetaR(1,:)) -ALL_r_R(1,:).*sin(ThetaR(1,:))]/a;

X2L(i_phi,:)=[ALL_r_L(2,:).*cos(ThetaL(2,:)) ALL_r_L(2,:).*cos(ThetaL(2,:))]/a;
Y2L(i_phi,:)=[ALL_r_L(2,:).*sin(ThetaL(2,:)) -ALL_r_L(2,:).*sin(ThetaL(2,:))]/a;
X2R(i_phi,:)=[ALL_r_R(2,:).*cos(ThetaR(2,:)) ALL_r_R(2,:).*cos(ThetaR(2,:))]/a;
Y2R(i_phi,:)=[ALL_r_R(2,:).*sin(ThetaR(2,:)) -ALL_r_R(2,:).*sin(ThetaR(2,:))]/a;

X3L(i_phi,:)=[ALL_r_L(3,:).*cos(ThetaL(3,:)) ALL_r_L(3,:).*cos(ThetaL(3,:))]/a;
Y3L(i_phi,:)=[ALL_r_L(3,:).*sin(ThetaL(3,:)) -ALL_r_L(3,:).*sin(ThetaL(3,:))]/a;
X3R(i_phi,:)=[ALL_r_R(3,:).*cos(ThetaR(3,:)) ALL_r_R(3,:).*cos(ThetaR(3,:))]/a;
Y3R(i_phi,:)=[ALL_r_R(3,:).*sin(ThetaR(3,:)) -ALL_r_R(3,:).*sin(ThetaR(3,:))]/a;
end

```

```

FontSize=20;
subplot(1,3,1)
h1=semilogx(X1L(1,:),Y1L(1,:), 'ob',X2L(1,:),Y2L(1,:), 'vr',X3L(1,:),Y3L(1,:), '^g',
X0(1,:),Y0(1,:), 'sk',Xf(1,:),Yf(1,:), 'sk',X1R(1,:),Y1R(1,:), 'ob',...
X2R(1,:),Y2R(1,:), 'vr',X3R(1,:),Y3R(1,:), '^g')
title('\phi = \pi/6')
xlabel('x/a', 'FontSize',FontSize);ylabel('y/a', 'FontSize',FontSize)
legend(['u* = ' num2str(fj(1))], ['u* = ' num2str(fj(2))],...
['u* = ' num2str(fj(3))], 'Location', 'West')
set(h1, 'LineWidth',2);
set(gca, 'FontSize',FontSize);
subplot(1,3,2)
h2=semilogx(X1L(2,:),Y1L(2,:), 'ob',X2L(2,:),Y2L(2,:), 'vr',X3L(2,:),Y3L(2,:), '^g',
X0(2,:),Y0(2,:), 'sk',Xf(2,:),Yf(2,:), 'sk',X1R(2,:),Y1R(2,:), 'ob',...
X2R(2,:),Y2R(2,:), 'vr',X3R(2,:),Y3R(2,:), '^g')
title('\phi = \pi/4')
xlabel('x/a', 'FontSize',FontSize);ylabel('y/a', 'FontSize',FontSize)
legend(['u* = ' num2str(fj(1))], ['u* = ' num2str(fj(2))],...
['u* = ' num2str(fj(3))], 'Location', 'West')
set(h2, 'LineWidth',2);
set(gca, 'FontSize',FontSize);
subplot(1,3,3)
h3=semilogx(X1L(3,:),Y1L(3,:), 'ob',X2L(3,:),Y2L(3,:), 'vr',X3L(3,:),Y3L(3,:), '^g',
X0(3,:),Y0(3,:), 'sk',Xf(3,:),Yf(3,:), 'sk',X1R(3,:),Y1R(3,:), 'ob',...
X2R(3,:),Y2R(3,:), 'vr',X3R(3,:),Y3R(3,:), '^g')
title('\phi = \pi/3')
xlabel('x/a', 'FontSize',FontSize);ylabel('y/a', 'FontSize',FontSize)
legend(['u* = ' num2str(fj(1))], ['u* = ' num2str(fj(2))],...
['u* = ' num2str(fj(3))], 'Location', 'West')
set(h3, 'LineWidth',2);
set(gca, 'FontSize',FontSize);

```

Fig. 7

```

clc
clear all
DataPoints=500;
b=1;
phi=pi/3;
a=b/(2*cos(phi)),alphaS=b/a
invM=((alphaS^4/2+2*alphaS^2-1)*phi-8/3*alphaS^3*sin(phi)+(alphaS^2-
2/3)*sin(2*phi)-sin(4*phi)/12)/((2-alphaS^2)*phi+sin(2*phi))
M=invM^(-1);
fmax=M*(1+alphaS-3/4*alphaS^2)*(alphaS/2-1)/(alphaS/2+1)
Theta=linspace(0,phi,DataPoints);r_out=2*a*cos(Theta);

fj=linspace(1e-1,0.99*fmax,DataPoints);

for j=1:length(fj)
    for i=1:DataPoints
        rPa=linspace(alphaS,r_out(i)/a,DataPoints);
        f_Theta=M*(rPa.^2-alphaS.^2).*(1-2*cos(Theta(i))./rPa);
        [error ind_i]=min((1-f_Theta/fj(j)).^2);
    end
end

```

```

        if error<1e-4
            rPa_j(j,i)=rPa(ind_i);Theta_j(j,i)=Theta(i);
        else
            rPa_j(j,i)=0;Theta_j(j,i)=0;
        end
    end
end

% plot(rPa_j(50,:).*cos(Theta_j(50,:)),rPa_j(50,:).*sin(Theta_j(50,:)),'sk')

for j=1:length(fj)
    Delta_r=rPa_j(j,:)*a-b;
    [MAXTheta MAXind]=max(Theta_j(j,:));
    indL_drj=find(Delta_r<Delta_r(MAXind));[ThetaL ind_L]=sort(Theta_j(j,indL_drj));
    r_Ll=rPa_j(j,indL_drj)*a;r_L=r_Ll(ind_L);
    i_L=0;
    for iii=1:length(ThetaL)
        if ThetaL(iii)
            i_L=i_L+1;
            ThetaLL(i_L)=ThetaL(iii);r_LL(i_L)=r_L(iii);
        elseif r_L(iii)
            i_L=i_L+1;
            ThetaLL(i_L)=ThetaL(iii);r_LL(i_L)=r_L(iii);
        end
    end
    indR_drj=find(Delta_r>Delta_r(MAXind));[ThetaR
ind_R]=sort(Theta_j(j,indR_drj));
    r_Rl=rPa_j(j,indR_drj)*a;r_R=r_Rl(ind_R);
    i_R=0;
    for jjj=1:length(ThetaR)
        if ThetaR(jjj)
            i_R=i_R+1;
            ThetaRR(i_R)=ThetaR(jjj);r_RR(i_R)=r_R(jjj);
        elseif r_R(jjj)
            i_R=i_R+1;
            ThetaRR(i_R)=ThetaR(jjj);r_RR(i_R)=r_R(jjj);
        end
    end
end

AcL(j)=-trapz(r_LL,r_LL)*trapz(ThetaLL,ones(1,length(ThetaLL)));
AcR(j)=-trapz(r_RR,r_RR)*trapz(ThetaRR,ones(1,length(ThetaRR)));
end
Ac=2*(AcR-AcL);
Act=(2*a^2-b^2)*phi+a^2*sin(2*phi);
coef=polyfit(fj,Ac/Ac(1),2)
FontSize=12;
h=plot(fj(50:end),Ac(50:end)/Act,'s',fj,polyval([-0.11 -0.17 1],fj))
xlabel('u*','FontSize',FontSize);ylabel('A_c/A_{ct}','FontSize',FontSize)
legend('Numerical evaluation','Cal.','Location','NorthEast')
set(h,'LineWidth',2);
set(gca,'FontSize',FontSize);

```

Fig. 8

```
clc
clear all
DataPoints=1e3;
b=1;
ALL_phi=[1/6 1/4 1/3]*pi;
for i_phi=1:length(ALL_phi)
    phi=ALL_phi(i_phi);
    [ALL_Theta(:,i_phi) ALL_E(:,i_phi)
ALL_fmax(i_phi)]=MoonRTDFun(phi,DataPoints,b);
end
x11=ALL_Theta(:,1);ind1=find(x11>0);x1=ALL_Theta(ind1,1);y1=ALL_E(ind1,1);
x22=ALL_Theta(:,2);ind2=find(x22>0);x2=ALL_Theta(ind2,2);y2=ALL_E(ind2,2);
x33=ALL_Theta(:,3);ind3=find(x33>0);x3=ALL_Theta(ind3,3);y3=ALL_E(ind3,3);

Theta01=round(1/ALL_fmax(1)*100)/100;x0=y1;
n1=(2-Theta01)/(1-Theta01);n1=round(n1*100)/100;
K1=(n1-1)*Theta01^(n1-2);K1=round(K1*100)/100;
y0=K1*Theta01./x1.^n1;
R0_Pearson=sum((x0-mean(x0)).*(y0-mean(y0)))/(sum((x0-mean(x0)).^2)*sum((y0-mean(y0)).^2))^0.5;
R2_1=round(R0_Pearson^2*100)/100
Theta_cal1=Theta01:0.01:max(x1);E_cal1=K1*Theta01./Theta_cal1.^n1;

Theta02=round(1/ALL_fmax(2)*100)/100;x0=y2;
n2=(2-Theta02)/(1-Theta02);n2=round(n2*100)/100;
K2=(n2-1)*Theta02^(n2-2);K2=round(K2*100)/100;
y0=K2*Theta02./x2.^n2;
R0_Pearson=sum((x0-mean(x0)).*(y0-mean(y0)))/(sum((x0-mean(x0)).^2)*sum((y0-mean(y0)).^2))^0.5;
R2_2=round(R0_Pearson^2*100)/100
Theta_cal2=Theta02:0.01:max(x2);E_cal2=K2*Theta02./Theta_cal2.^n2;

Theta03=round(1/ALL_fmax(3)*100)/100;x0=y3;
n3=(2-Theta03)/(1-Theta03);n3=round(n3*100)/100;
K3=(n3-1)*Theta03^(n3-2);K3=round(K3*100)/100;
y0=K3*Theta03./x3.^n3;
R0_Pearson=sum((x0-mean(x0)).*(y0-mean(y0)))/(sum((x0-mean(x0)).^2)*sum((y0-mean(y0)).^2))^0.5;
R2_3=round(R0_Pearson^2*100)/100
Theta_cal3=Theta03:0.01:max(x3);E_cal3=K3*Theta03./Theta_cal3.^n3;

FontSize=24;
subplot(2,2,[1 3])
h=loglog(x1,y1,'sb',x2,y2,'or',x3,y3,'*g',Theta_cal1,E_cal1,'b',...
    Theta_cal2,E_cal2,':r',Theta_cal3,E_cal3,'--g')
xlabel('\theta','FontSize',FontSize);ylabel('E(\theta)','FontSize',FontSize)
legend('\phi = \pi/6','\phi = \pi/4','\phi = \pi/3',...
    ['\phi = \pi/6; \theta_{min} = ' num2str(Theta01) '; K = ' num2str(K1) ...
    '; n = ' num2str(n1)],['\phi = \pi/4; \theta_{min} = ' ...
    num2str(Theta02) '; K = ' num2str(K2) '; n = ' num2str(n2)], ...
    ['\phi = \pi/3; \theta_{min} = ' num2str(Theta03) '; K = ' num2str(K3) ...
    '; n = ' num2str(n3)],'Location','NorthEast')
set(h,'LineWidth',2);
```



```

set(gca,'FontSize',FontSize);
subplot(2,2,2)
h=plot(x1,y1,'sb',x2,y2,'or',x3,y3,'*g',Theta_cal1,E_cal1,'b',...
    Theta_cal2,E_cal2,':r',Theta_cal3,E_cal3,'--g')
xlabel('\theta','FontSize',FontSize);ylabel('E(\theta)','FontSize',FontSize)
xlim([0 2])
set(h,'LineWidth',2);
set(gca,'FontSize',FontSize);

function [Theta_RTD E fmax]=MoonRTDFun(phi,DataPoints,b)
a=b/(2*cos(phi));
alphaS=b/a;
invM=((alphaS^4/2+2*alphaS^2-1)*phi-8/3*alphaS^3*sin(phi)+...
    (alphaS^2-2/3)*sin(2*phi)-sin(4*phi)/12)/((2-alphaS^2)*phi+sin(2*phi));
M=invM^(-1);
fmax=M*(1+alphaS-3/4*alphaS^2)*(alphaS/2-1)/(alphaS/2+1);
Theta=linspace(0,phi,DataPoints);r_out=2*a*cos(Theta);

fj=linspace(1e-2,0.99*fmax,DataPoints);

for j=1:length(fj)
    for i=1:DataPoints
        rPa=linspace(alphaS,r_out(i)/a,DataPoints);
        f_Theta=M*(rPa.^2-alphaS.^2).*(1-2*cos(Theta(i))./rPa);
        [error ind_i]=min((1-f_Theta/fj(j)).^2);
        if error<1e-3
            rPa_j(j,i)=rPa(ind_i);Theta_j(j,i)=Theta(i);
        else
            rPa_j(j,i)=0;Theta_j(j,i)=0;
        end
    end
end

G=abs(-0.22*fj-0.17);
Theta_RTD=1./fj;
E=Theta_RTD.^(-3).*G;

```

Fig. 9

```

clc
clear all
a=1;
bPa=0.1:0.1:10;
ALL_Theta=zeros(2000,length(bPa));
ALL_E=ALL_Theta;
for nb=1:length(bPa)
    b=bPa(nb)*a;

    n_sum=21;
    z=linspace(-a,a,2e3);
    y=linspace(-b,b,2e3);
    sumterm1=zeros(length(y),length(z));sumterm2=0;sumterm3=0;
    for n=1:2:n_sum
        sumterm1=sumterm1+1/n^3*(-1)^((n-1)/2)*(1-cosh(n*pi*y/2/a)/...
            cosh(n*pi*b/a/2)).*cos(n*pi*z/2/a);
    end
end

```

```

        sumterm2=sumterm2+1/n^5*tanh(n*pi*b/a/2);
        sumterm3=sumterm3+1/n^3*(-1)^((n-1)/2)*(1-1/cosh(n*pi*b/a/2));
    end
    f=48/pi^3*sumterm1/(1-192/pi^5*a/b*sumterm2);
    f_max=48/pi^3*sumterm3/(1-192/pi^5*a/b*sumterm2);
    ALL_fmax(nb)=f_max;
    Theta_min=1/f_max;
    n=(2-Theta_min)/(1-Theta_min);
    ALL_n(nb)=n;
    ALL_K(nb)=(n-1)*Theta_min^(n-2);

end
FontSize=24;
FontSize2=17;

subplot(2,2,1)
h1=plot(bPa,ALL_fmax)
title(' (a) ')
xlabel('b/a','FontSize',FontSize);ylabel('u_{max}','FontSize',FontSize)
set(h1,'LineWidth',2);
set(gca,'FontSize',FontSize2);
subplot(2,2,2)
h2=plot(bPa,1./ALL_fmax)
title(' (b) ')
xlabel('b/a','FontSize',FontSize);ylabel('\theta_{min}','FontSize',FontSize)
set(h2,'LineWidth',2);
set(gca,'FontSize',FontSize2);
subplot(2,2,3)
h3=plot(bPa,ALL_K)
title(' (c) ')
xlabel('b/a','FontSize',FontSize);ylabel('K','FontSize',FontSize)
set(h3,'LineWidth',2);
set(gca,'FontSize',FontSize2);
subplot(2,2,4)
h3=plot(bPa,ALL_n)
title(' (d) ')
xlabel('b/a','FontSize',FontSize);ylabel('n','FontSize',FontSize)
set(h3,'LineWidth',2);
set(gca,'FontSize',FontSize2);

```

Fig. 10

```

clc
clear all
bPa=0.5;
etha_z=linspace(-0.5,0.5,100);    %z/2a
etha_y=etha_z;    %y/2b
[Etha_Z,Etha_Y] = meshgrid(etha_z,etha_y);
sumterm1=zeros(length(Etha_Y),length(Etha_Z));sumterm2=0;sumterm3=0;
for n=1:2:1001
    sumterm1=sumterm1+1/n^3*(-1)^((n-1)/2)*(1-cosh(n*pi*Etha_Y*bPa)/...
        cosh(n*pi*bPa/2)).*cos(n*pi*Etha_Z);
    sumterm2=sumterm2+1/n^5*tanh(n*pi*bPa/2);
    sumterm3=sumterm3+1/n^3*(-1)^((n-1)/2)*(1-1/cosh(n*pi*bPa/2));
end
f=48/pi^3*sumterm1/(1-192/pi^5/bPa*sumterm2);

```

```

f_max=48/pi^3*sumterm3/(1-192/pi^5/bPa*sumterm2)

FontSize=12;
contour(Etha_Z,Etha_Y,f,'LineWidth',4,'ShowText','on',...
'LabelSpacing',1000,'TextList',[0.4 0.8 1.2 1.8])
xlabel('z/(2a)','FontSize',FontSize);ylabel('y/(2b)','FontSize',FontSize)
set(gca,'FontSize',FontSize);

```

Fig. 11

```

clc
clear all
a=1;bPa=0.5;
DataPoints=2e3;
ALL_Theta=zeros(DataPoints,length(bPa));
ALL_E=ALL_Theta;
b=bPa*a;

n_sum=21;
z=linspace(-a,a,DataPoints);
y=linspace(-b,b,DataPoints);
sumterm1=zeros(length(y),length(z));sumterm2=0;sumterm3=0;
for n=1:2:n_sum
    sumterm1=sumterm1+1/n^3*(-1)^((n-1)/2)*(1-cosh(n*pi*Etha_Y*bPa)/...
        cosh(n*pi*bPa/2)).*cos(n*pi*Etha_Z);
    sumterm2=sumterm2+1/n^5*tanh(n*pi*b/a/2);
    sumterm3=sumterm3+1/n^3*(-1)^((n-1)/2)*(1-1/cosh(n*pi*b/a/2));
end
f=48/pi^3*sumterm1/(1-192/pi^5*a/b*sumterm2);
f_max=48/pi^3*sumterm3/(1-192/pi^5*a/b*sumterm2);
Selected_f=1e-2:1e-2:f_max;

for i=1:length(Selected_f)
    [ind_Z ind_Y]=find((1-f/Selected_f(i)).^2<1e-3);
    zi=z(ind_Z);yi=y(ind_Y);
    ind_p=find(zi>0);zii=zi(ind_p);yii=yi(ind_p);
    Delta_y=diff(yii);
    HalfArea(i)=sum(Delta_y.*zii(1:end-1));
end
All_Ac=2*HalfArea;
dAc=diff(All_Ac);
Act=4*a*b;
FontSize=12;
h=plot(Selected_f,All_Ac/Act)
xlabel('u*', 'FontSize',FontSize);ylabel('A_c/A_{ct}','FontSize',FontSize)
set(h,'LineWidth',2);
set(gca,'FontSize',FontSize);

```

Fig. 12

```

clc
clear all
a=1;bPa=[0.5 1 1.5 2];
DataPoints=2e3;
ALL_Theta=zeros(DataPoints,length(bPa));

```

```

ALL_E=ALL_Theta;
for nb=1:length(bPa)
    clear theta E g dr Selected_f dAc dy dz All_Ac ALL_y ALL_z HalfArea
    b=bPa(nb)*a;

    n_sum=21;
    z=linspace(-a,a,DataPoints);
    y=linspace(-b,b,DataPoints);
    sumterm1=zeros(length(y),length(z));sumterm2=0;sumterm3=0;
    for n=1:2:n_sum
        sumterm1=sumterm1+1/n^3*(-1)^((n-1)/2)*(1-cosh(n*pi*y/2/a)/...
            cosh(n*pi*b/a/2)).*cos(n*pi*z/2/a);
        sumterm2=sumterm2+1/n^5*tanh(n*pi*b/a/2);
        sumterm3=sumterm3+1/n^3*(-1)^((n-1)/2)*(1-1/cosh(n*pi*b/a/2));
    end
    f=48/pi^3*sumterm1/(1-192/pi^5*a/b*sumterm2);
    f_max=48/pi^3*sumterm3/(1-192/pi^5*a/b*sumterm2);
    ALL_fmax(nb)=f_max;
    Selected_f=1e-2:1e-2:f_max;

    for i=1:length(Selected_f)
        [ind_Z ind_Y]=find((1-f/Selected_f(i)).^2<1e-3);
        zi=z(ind_Z);yi=y(ind_Y);
        ind_p=find(zi>0);zii=zi(ind_p);yii=yi(ind_p);
        Delta_y=diff(yii);
        HalfArea(i)=sum(Delta_y.*zii(1:end-1));
    end
    All_Ac=2*HalfArea;
    dAc=diff(All_Ac);
    G=abs(dAc/All_Ac(1)./diff(Selected_f));
    theta=1./(Selected_f(1:end-1)+diff(Selected_f));
    E=theta.^(-3).*G;
    [Sorted_theta ind_theta]=sort(theta);Sorted_E=E(ind_theta);
    Area_E(nb)=trapz(Sorted_theta,Sorted_E)

    ALL_Theta(1:length(Sorted_theta),nb)=Sorted_theta;
    ALL_E(1:length(Sorted_theta),nb)=Sorted_E;
end
x11=ALL_Theta(:,1);ind1=find(x11>0);x1=ALL_Theta(ind1,1);y1=ALL_E(ind1,1);
x22=ALL_Theta(:,2);ind2=find(x22>0);x2=ALL_Theta(ind2,2);y2=ALL_E(ind2,2);
x33=ALL_Theta(:,3);ind3=find(x33>0);x3=ALL_Theta(ind3,3);y3=ALL_E(ind3,3);
x44=ALL_Theta(:,4);ind4=find(x44>0);x4=ALL_Theta(ind4,4);y4=ALL_E(ind4,4);

Theta01=round(1/ALL_fmax(1)*100)/100;x0=y1;
n1=(2-Theta01)/(1-Theta01);n1=round(n1*100)/100;
K1=(n1-1)*Theta01^(n1-2);K1=round(K1*100)/100;
y0=K1*Theta01./x1.^n1;
R0_Pearson=sum((x0-mean(x0)).*(y0-mean(y0)))/(sum((x0-mean(x0)).^2)*sum((y0-mean(y0)).^2))^0.5;
R2_1=round(R0_Pearson^2*100)/100
Theta_call1=Theta01:0.01:max(x1);E_call1=K1*Theta01./Theta_call1.^n1;

Theta02=round(1/ALL_fmax(2)*100)/100;x0=y2;
n2=(2-Theta02)/(1-Theta02);n2=round(n2*100)/100;

```

```

K2=(n2-1)*Theta02^(n2-2);K2=round(K2*100)/100;
y0=K2*Theta02./x2.^n2;
R0_Pearson=sum((x0-mean(x0)).*(y0-mean(y0)))/(sum((x0-mean(x0)).^2)*sum((y0-mean(y0)).^2))^0.5;
R2_2=round(R0_Pearson^2*100)/100
Theta_cal2=Theta02:0.01:max(x2);E_cal2=K2*Theta02./Theta_cal2.^n2;

Theta03=round(1/ALL_fmax(3)*100)/100;x0=y3;
n3=(2-Theta03)/(1-Theta03);n3=round(n3*100)/100;
K3=(n3-1)*Theta03^(n3-2);K3=round(K3*100)/100;
y0=K3*Theta03./x3.^n3;
R0_Pearson=sum((x0-mean(x0)).*(y0-mean(y0)))/(sum((x0-mean(x0)).^2)*sum((y0-mean(y0)).^2))^0.5;
R2_3=round(R0_Pearson^2*100)/100
Theta_cal3=Theta03:0.01:max(x3);E_cal3=K3*Theta03./Theta_cal3.^n3;

Theta04=round(1/ALL_fmax(4)*100)/100;x0=y4;
n4=(2-Theta04)/(1-Theta04);n4=round(n4*100)/100;
K4=(n4-1)*Theta04^(n4-2);K4=round(K4*100)/100;
y0=K4*Theta04./x4.^n4;
R0_Pearson=sum((x0-mean(x0)).*(y0-mean(y0)))/(sum((x0-mean(x0)).^2)*sum((y0-mean(y0)).^2))^0.5;
R2_4=round(R0_Pearson^2*100)/100
Theta_cal4=Theta04:0.01:max(x4);E_cal4=K4*Theta04./Theta_cal4.^n4;

FontSize=17;
subplot(2,2,[1 3])
h=loglog(x1,y1,'sb',x2,y2,'or',x3,y3,'*g',x4,y4,'+m',Theta_cal1,E_cal1,'b',...
    Theta_cal2,E_cal2,':r',Theta_cal3,E_cal3,'--g',Theta_cal4,E_cal4,'-.m')
xlabel('\theta','FontSize',FontSize);ylabel('E(\theta)','FontSize',FontSize)
legend(['b/a = ' num2str(bPa(1))],[ 'b/a = ' num2str(bPa(2))],...
    ['b/a = ' num2str(bPa(3))],[ 'b/a = ' num2str(bPa(4))],...
    ['b/a = ' num2str(bPa(1)) ';' \theta_{min} = ' num2str(Theta01) ...
    ';' K = ' num2str(K1) ';' n = ' num2str(n1)], ...
    ['b/a = ' num2str(bPa(2)) ';' \theta_{min} = ' num2str(Theta02) ...
    ';' K = ' num2str(K2) ';' n = ' num2str(n2)], ...
    ['b/a = ' num2str(bPa(3)) ';' \theta_{min} = ' num2str(Theta03) ...
    ';' K = ' num2str(K3) ';' n = ' num2str(n3)], ...
    ['b/a = ' num2str(bPa(4)) ';' \theta_{min} = ' num2str(Theta04) ...
    ';' K = ' num2str(K4) ';' n = ' num2str(n4)], ...
    'Location','NorthEast')
set(h,'LineWidth',2);
set(gca,'FontSize',FontSize);
subplot(2,2,2)
h2=plot(x1,y1,'sb',x2,y2,'or',x3,y3,'*g',x4,y4,'+m',Theta_cal1,E_cal1,'b',...
    Theta_cal2,E_cal2,':r',Theta_cal3,E_cal3,'--g',Theta_cal4,E_cal4,'-.m')
xlabel('\theta','FontSize',FontSize);ylabel('E(\theta)','FontSize',FontSize)
xlim([0 2]);
set(h2,'LineWidth',2);
set(gca,'FontSize',FontSize);

```

Fig. 13

```

clc
clear all
a=14;
b=2.46;

```

```

n_sum=21;
z=linspace(-a,a,2e3);
y=linspace(-b,b,2e3);
sumterm1=zeros(length(y),length(z));sumterm2=0;sumterm3=0;
for n=1:2:n_sum
    sumterm1=sumterm1+1/n^3*(-1)^( (n-1)/2)*(1-cosh(n*pi*y/2/a)/...
        cosh(n*pi*b/a/2)).*cos(n*pi*z/2/a);
    sumterm2=sumterm2+1/n^5*tanh(n*pi*b/a/2);
    sumterm3=sumterm3+1/n^3*(-1)^( (n-1)/2)*(1-1/cosh(n*pi*b/a/2));
end
f=48/pi^3*sumterm1/(1-192/pi^5*a/b*sumterm2);
f_max=48/pi^3*sumterm3/(1-192/pi^5*a/b*sumterm2);
Theta_min=1/f_max
n=(2-Theta_min)/(1-Theta_min)
K=(n-1)*Theta_min^(n-2)
Theta_min2=exp(1/n*log(Theta_min/2.5))
Theta=Theta_min:1e-2:3.5;
E=K*Theta_min./Theta.^n;

% AA1=imread('Exp.png');
% AA=imresize(AA1,3);
% subplot(2,1,1)
% imshow(AA)
% subplot(2,1,2)

FontSize=28;
h=plot([0 Theta_min Theta],[0 0 E],Theta_min2*ones(1,length(E)),E,':k')
xlabel('\Theta','FontSize',FontSize);ylabel('E(\Theta)','FontSize',FontSize)
axis([0 3.5 0 5.5]);
xticks([0 0.5 0.7 1:0.5:3.5])
xticklabels({'0','0.5','0.7','1','1.5','2','2.5','3','3.5'})
set(h,'LineWidth',2);
set(gca,'FontSize',FontSize);

```

Fig. 14

```

clc
clear all
%*****Falling Film Flow
Theta_minFFF=2/3;
ThetaFFF=linspace(Theta_minFFF+0.01,2,100);
nFFF=(2-Theta_minFFF)/(1-Theta_minFFF)
K_FFF=Theta_minFFF^(Theta_minFFF/(1-Theta_minFFF))/(1-Theta_minFFF)
% K_FFF=(nFFF-1)*Theta_minFFF^(nFFF-2)
E_FFF=1/2*(1-Theta_minFFF./ThetaFFF).^(-1/2).*Theta_minFFF./ThetaFFF.^3;
E_Cal_FFF=K_FFF*Theta_minFFF./ThetaFFF.^nFFF;
%*****Annular
alpha=0.3;
lambda=((1-alpha^2)/2/log(1/alpha))^0.5;
x=linspace(alpha+0.01,lambda-0.01,100);
ALL_xbar=lambda:0.01:1;
Theta=1/2*((1+alpha^2-2*lambda^2)./(1-x.^2+2*lambda^2*log(x)));
Theta_min=1/2*((1+alpha^2-2*lambda^2)/(1-lambda^2+2*lambda^2*log(lambda)))

```

```

for i=1:length(x)
    Error=abs(ALL_xbar.^2-2*lambda^2*log(ALL_xbar)-x(i)^2+2*lambda^2*log(x(i)));
    [err ind]=min(Error);
    xbar(i)=ALL_xbar(ind);
end
E=4*(1-xbar.^2+2*lambda^2*log(xbar)).^2.*(xbar-xbar.^3- ...
    x.^2.*(xbar.^2-lambda^2).*(1-x.^3+2*lambda^2*log(x))./xbar./ ...
    (x.^2-lambda^2)+2*xbar*lambda^2.*log(xbar))./((1-alpha^2)* ...
    (1+alpha^2-2*lambda^2)^2*(xbar-lambda^2./xbar));

n=(2-Theta_min)/(1-Theta_min)
K=Theta_min^(Theta_min/(1-Theta_min))/(1-Theta_min)
E_Cal=K*Theta_min./Theta.^n;

FontSize=15;
h=plot(ThetaFFF,E_FFF,'b',ThetaFFF,E_Cal_FFF,':b',Theta,E,'sk',Theta,E_Cal,':k')
legend('Precise:Film','Proposed Model:Film','Precise:Annulus',...
    'Proposed Model:Annulus','Location','NorthEast')
xlim([Theta_min 1])
ylim([0 10])
xlabel('\Theta','FontSize',FontSize);ylabel('E(\Theta)','FontSize',FontSize)
set(h,'LineWidth',2);
set(gca,'FontSize',FontSize);

```