Cattaneo-Christov Heat Flux Model of Eyring Powell Fluid Along with Convective Boundary Conditions

Bilal Ashraf, Muhammad**
Department of Mathematics, COMSATS University Islamabad, Islamabad 44000, PAKISTAN

Sharma, Sapna; Aneja, Madhu
School of Mathematics, Thapar Institute of Engineering & Technology, Patiala-147004, INDIA

ABSTRACT: Heat and mass transfer effects in three-dimensional mixed convection flow of Eyring Powell fluid over an exponentially stretching surface with convective boundary conditions are inspected. Cattaneo-Christov Heat Flux model is a modified version of the classical Fourier’s law that takes into account the interesting aspect of thermal relaxation time. First-order chemical reaction effects are also taken into account. Similarity transformations are invoked to reduce the leading boundary layer partial differential equations into the ordinary differential equations. The nonlinear, coupled ordinary differential with boundary conditions has been analyzed numerically by using the Finite Element Method.

KEYWORDS: Convective boundary conditions; Cattaneo-Christov heat flux model; Eyring Powell fluid; Exponential stretching sheet; Finite Element Method.

INTRODUCTION

Glass blowing, continuous casting of metals, and spinning of fibers involve the flow due to a stretching surface in an ambient fluid. The quality of the final sheeting material, as well as the cost of production is affected by the speed of collection and the heat transfer rate. Also, in several engineering processes, materials manufactured by extrusion processes and heat-treated materials traveling between a feed roll and a wind-up roll-on convey belts possess the characteristic of a moving continuous surface. After the initial contribution of Crane [1], various researchers extended the flow over a stretching surface in directions of Newtonian and non-Newtonian fluid models under different geometries. Mechanics of non-Newtonian fluid flows present a special challenge to engineers, physicists, and mathematicians. Because of the complexity of these fluids, there is not a single constitutive equation that exhibits all properties of such non-Newtonian fluids. In the process, a number of non-Newtonian fluid models have been proposed. In the literature, the vast majority of non-Newtonian fluid are concerned of the types, e.g., the power-law [2], viscoelastic fluid [3], grade two or three [4], Maxwell fluid [5]. Flow problems involve the exponential dependence of the similarity variables as the stretching velocity and temperature distribution are presented by exponentially stretching the surface. Mostly the researchers...
presented the solutions for two-dimensional exponentially stretching flows. Thermal boundary layer over an exponentially stretching continuous surface with magnetic field effect was investigated numerically by Al-Odat et al. [6]. Analytical solutions of nanofluid flow over an exponential stretching surface were presented by Nadeem and Lee [7]. Heat transfer analysis over an exponentially shrinking surface by applying shooting method was investigated by Bhattacharyya [8]. Heat transfer over a permeable stretching surface for a Casson fluid was studied by Mukhopadhyay et al. [9]. After that, Three-dimensional viscous flow in presence of heat transfer over an exponentially stretching surface is studied by Liu et al. [10]. Recently Ashraf et al. [11] extended the flow of Liu et al. [10] by considering the viscoelastic fluid.

Many investigators stressed on the magnetohydrodynamic flow of an electrically conducting fluid because of numerous applications in metallurgical industry such as in drawing, annealing, in the purification of molten metals from non-metallic inclusions, electromagnetic pumps, MHD generators etc. Several studies have been presented by the authors in presence of transverse magnetic field over a stretching surface. Magnetohydrodynamic boundary layer flow with variable viscosity over a stretching surface was presented by Mukhopadhyay et al. [12]. Motsa et al. [13] obtained the solutions for upper convected Maxwell fluid over porous stretching sheet in presence of magnetic field by using successive Taylor series linearization method. Rashidi and Erfani [14] studied the stagnation-point flow in porous media with magnetic field.

Heat conduction is a widespread nature phenomenon occurring in many fields [16-21]. The classical heat conduction equation is based on the Fourier law. The classical Fourier law has been highly successful in a very wide range of engineering applications. However, one unphysical property is that it issues an infinite velocity of propagation, i.e., any initial disturbance is felt instantly throughout the whole of the medium. In order to overcome this problem, Cattaneo [22] modified the Fourier law by taking the finite velocity of propagation into account. This extension turns the diffusion equation from a parabolic equation to a hyperbolic one. In recent years, many researchers devoted themselves to the study of heat conduction with Cattaneo model. Christov and Jordan [23] pointed out that the usual form of Maxwell-Cattaneo leads to a paradoxical result if the body is in motion, the governing equation cannot be reduced to a single transport equation in the multidimensional case. Qi and Gao [24] studied the initial-boundary value problem of general-ized Cattaneo equation, the exact solution was obtained in series forms in terms of the H-function by utilizing the Laplace transform. Compte and Metzler [25] generalized the Cattaneo equation to study the anomalous transport, three possible generalizations were pro-posed which can be used to describe continuous time random walks, non-local transport theory, and delayed flux-force relation. Straughan [26] investigated convection effects in the horizontal layer of incompressible fluid over a flat plate utilizing Cattaneo--Christov heat flux. Tibullo and Zampoli [27] proved existence and uniqueness of solutions for equations governing heat transfer in incompressible fluids through Cattaneo--Christov theory. Haddad [28] examined thermal instability in Brinkman porous media considering non-Fourier heat flux.

Thus purpose of present research is to study the three dimensional flow of Eyring Powell fluid with convective boundary conditions by using Finite Element Method [29-32]. The novelty. Many authors investigated three dimensional flow of Eyring Powell fluid over a stretching sheet with different boundary conditions [33, 34]. The novelty of the problem is to investigate Cattaneo-Christov heat flux model for three dimensional flow of Eyring Powell fluid. Here the temperature and thermal boundary layer thickness are lower for Cattaneo-Christov heat flux model in comparison to classical Fourier's law of heat conduction. Impacts of embedding parameters on the flow and temperature fields are examined graphically. Numerical values of skin-friction coefficient, transversal skin-friction coefficient and local Nusselt number for sundry variables are obtained and discussed.

**THEORETICAL SECTION**

**Mathematical modeling**

We consider the three-dimensional boundary layer flow of Eyring Powell nanofluid by an exponentially stretching surface. The surface coincides with the plane \( z = 0 \) and the flow is confined in the region \( z > 0 \). Heat and mass transfer effects are taken into account. Effects of Brownian motion and thermophoresis are considered. The Cauchy stress tensor \( T \) for an Eyring Powell Fluid can be written as [33]:

\[
T = pI + \tau_{ij}
\]  

(1)
\[
\rho_a = -V^p + V \cdot \tau_{ij}
\]

where extra stress tensor \( \tau_{ij} \) is given by

\[
\tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} + \frac{1}{\beta} \sinh \left( \frac{1}{c} \frac{\partial u_i}{\partial x_j} \right)
\]

in which \( \beta \) and \( c \) are the Eyring Powell fluid characteristics, \( p \) is the pressure and \( I \) is the identity tensor. Taking

\[
\sinh^{-1} \left( \frac{1}{c} \frac{\partial u_i}{\partial x_j} \right) = \frac{1}{\beta} \left( \frac{\partial u_i}{\partial x_j} \right) + \frac{1}{6} \left( \frac{\partial u_i}{\partial x_j} \right)^3 < 1
\]

The governing equations for the three-dimensional flow can be put into the form:

\[
\dot{u} \frac{\partial u}{\partial x} + \dot{v} \frac{\partial u}{\partial y} + \dot{w} \frac{\partial u}{\partial z} = 0
\]

\[
\dot{u} \frac{\partial u}{\partial x} + \dot{v} \frac{\partial v}{\partial y} + \dot{w} \frac{\partial v}{\partial z} = \left( 9 + \frac{1}{\beta c \rho} \right) \frac{\partial^2 u}{\partial x^2} - \frac{1}{2 \beta c \rho} \frac{\sigma^2 B_0^2}{\partial z^2} - \frac{1}{\rho} u + g \beta \left( T - T_a \right) + \beta \left( C - C_a \right)
\]

\[
\dot{u} \frac{\partial u}{\partial x} + \dot{v} \frac{\partial v}{\partial y} + \dot{w} \frac{\partial v}{\partial z} = \left( 9 + \frac{1}{\beta c \rho} \right) \frac{\partial^2 v}{\partial y^2} - \frac{1}{2 \beta c \rho} \frac{\sigma^2 B_0^2}{\partial z^2} - \frac{1}{\rho} u + g \beta \left( T - T_a \right) + \beta \left( C - C_a \right)
\]

\[
\dot{u} \frac{\partial u}{\partial x} + \dot{v} \frac{\partial v}{\partial y} + \dot{w} \frac{\partial w}{\partial z} = D \frac{\partial^2 w}{\partial z^2} + D_v \frac{\partial^2 T}{\partial z^2} - \frac{K}{2} \frac{\partial^2 T}{\partial z^2}
\]

\[
\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} - \frac{K}{2} \frac{\partial^2 T}{\partial z^2}
\]

The corresponding boundary conditions are [15]:

\[
u \rightarrow 0, \quad v \rightarrow 0, \quad T \rightarrow T_{\infty}, \quad C \rightarrow C_{\infty}
\]

The velocities, temperature and concentration at the wall are [33]:

\[
U_w = U_{w e}^{x+y} \quad V_w = V_{w e}^{x+y} \quad w_0 = W_{w e}^{x+y}
\]

The following transformations are used to convert the governing equations of the flow problem into non-dimensionalized form [33]:

\[
u = U_{w e}^{x+y} f(x) \quad v = V_{w e}^{x+y} g(x)
\]

\[
T = T_{w e}^{x+y} + 2L z \quad \theta = \frac{U_{w e}^{x+y}}{2L} e^{2L z}
\]

The Eq. (5) is automatically satisfied with above mentioned transformations and Eqs (6-11) in non-dimensionalized form are:

\[
\left( 1 + e \right) T' + \left( f + g \right) T' - 2 \left( f' + g \right) T' = \frac{C}{2} \left( f' + g \right) T' - 2 \left( f' + g \right) T'
\]

\[
e^{\delta (f')^2} T' - M f' + \lambda c \theta + \lambda c \varphi = 0
\]


\[(1+\varepsilon)g^*+(f+g)g^*−2(f^*+g^*)g^*−2(f^*+g^*)g^*\]  
\[\varepsilon \delta (g^*)^2 g^*−Mg^*=0\]  
\[
\frac{1}{Pr} \frac{\partial \theta^*}{\partial t} + \frac{\gamma}{2} (A \theta (f+g)(f^*+g^*)−
A \theta (f^*+g^*)+\theta^*(f+g)=0
\]

\[\phi^*+\frac{Sc (f+g)\phi^*−Sc B (f^*+g^*)\phi+\frac{Nt}{Nb}}{0^*−Sc \phi}=0\]  

The boundary conditions (12) in non-dimensionalized form are:

\[f = 0, \ g = 0, \ f' = 1, \ g' = \varepsilon, \ \theta' = -\gamma_1 (1-0(0))\]  
\[\phi' = -\gamma_2 (1-\phi(0)) \text{ at } \eta = 0 \text{ and} \]
\[f' \to 0, \ g' \to 0, \ \theta' \to 0, \ \phi' \to 0 \text{ as } \eta \to \infty\]  

The various non-dimensional number mentioned in the above equations are mentioned in nomenclature and are defined as:

\[\varepsilon = \frac{1}{\mu \beta_x} \text{, } \delta = \frac{U^*_{Re} L}{2 \sigma^2 \beta_x^3} \text{, } Pr = \frac{\beta}{\alpha} \]  

\[N_b = \frac{\rho \varepsilon^2 D_B (C_u - C_w)}{\rho \varepsilon \beta}, \ k = \frac{k_1}{U_0}, \ M = \frac{\alpha \rho}{B_0^2} \]
\[N_t = \frac{\rho \varepsilon^2 D_B (T_w - T_u)}{\rho c T_0 \theta}, \ Sc = \frac{\beta}{D_B}, \ \omega = \frac{V_{Re}}{U_0} \]
\[\gamma_1 = \frac{h}{k}, \ \gamma_2 = \frac{h}{D}, \ \gamma = \frac{\lambda_1 U_{Re} e}{L} \]

The quantity of physical interest are local Nusselt number and local Sherwood number which are defined as

\[\frac{Nu}{Re_1^{1/2}} = \frac{x}{2L} \theta^*(0), \ \frac{Sh}{Re_1^{1/2}} = \frac{x}{2L} \phi^*(0)\]  

Where \(Re_1 = \frac{\rho_{\beta} x}{\beta \sigma^2 \beta_x^3}\) is the local Reynolds number.

**Solution Approach**

The system of nonlinear ordinary differential Eqs (15-18) subjected to boundary conditions (19) has been solved numerically by finite element method. Momentum Eqs (15, 16) is third order coupled equations. So, these equations are solved in the coupled form. In order to convert third order equation into second order, we assume:

\[f' = h \]
\[g' = 1 \]
\[\phi' = -\gamma_1 (1-\phi(0)) \text{ at } \eta = 0 \text{ and} \]
\[f' \to 0, \ g' \to 0, \ \phi' \to 0 \text{ as } \eta \to \infty\]  

\[\frac{1}{Pr} \frac{\partial \theta^*}{\partial t} + \frac{\gamma}{2} (A \theta (f+g)(h^*+1')−A \theta (2+A)(f'+g')^2 + \gamma_1 \gamma_2 \theta^*\phi^* + \lambda_1 \theta^* \phi^* = 0)\]  

\[\phi^*+\frac{Sc (f+g)\phi^*−Sc B (f^*+g^*)\phi+\frac{Nt}{Nb}}{0^*−Sc \phi}=0\]  

\[f = 0, \ g = 0, \ h = 1, \ 1 = \alpha, \ \theta' = -\gamma_1 (1-0(0)) \]  
\[\phi' = -\gamma_2 (1-\phi(0)) \text{ at } \eta = 0 \text{ and} \]
\[f' \to 0, \ g' \to 0, \ \phi' \to 0 \text{ as } \eta \to \infty\]  

A grid independence test is performed by taking uniform grid of different sizes such as 80, 100, 120, 140 nodes. The results are not much affected by varying grid size. So, results are obtained by taking 120 nodes of uniform size. Also results has been validated with previous obtained results by Bilal et al. [33] by considering special case in which thermal relaxation time \(\gamma=0\) and is presented in Fig. 1.

**RESULTS AND DISCUSSION**

The nonlinear, coupled ordinary differential Eqs. (15-18) with boundary conditions (19) has been analyzed numerically using FEM. The influence of important parameters such as Eyring Powell fluid parameters \(\varepsilon, \delta\), relaxation time for heat flux \(\gamma\), heat and mass transfer Biot number \(\gamma_1\) and \(\gamma_2\), Prandtl number \(Pr\), Brownian motion parameter \(Nb\). Thermophoresis parameter \(Nt\) are analyzed on velocity profile, temperature profile, concentration
Fig 1: Validation of results with previous studies.

Fig 2: Effect of $\delta$ on velocity profile.

Fig 3: Effect of $\eta$ on velocity profile.

Fig 4: Effect of $N_b$ on temperature profile.

profile graphically. Fig. 2 describes the velocity profile for different values of $\epsilon$. Since $\epsilon = \frac{1}{\rho \beta c}$ so by increasing $\epsilon$ viscosity of fluid decreases, which causes increase in velocity. Fig. 3 reflects the effect of $\delta$ on velocity profile. It is depicted that velocity decreases with increase in $\delta$. This is due to the fact that viscosity of fluid increases with increase in $\delta$ which causes decrease in velocity profile.

Figs. 4-8 show the variation of temperature field $\theta(\eta)$ for different values of Brownian motion parameter $N_b$, thermophoresis parameter $N_t$, relaxation time for heat flux $\gamma$, heat transfer Biot number $\gamma_1$, Prandtl number $Pr$. Fig. 4 show the variation of temperature profile and boundary layer thickness for increasing values of $N_b$. Random motion of fluid increases by increasing the Brownian motion parameter $N_b$. Thus fluid particles frequently strike with each other which produces more heat so it raises temperature.

Fig. 5 shows characteristics of thermophoresis parameter $N_t$ on temperature profile. Thermophoresis is a mechanism in which small particles are pulled away from hot surface to cold surface. Due to this transportation temperature of the fluid rises. This is also happened because additional heat is generated due to the interaction of nano-particles with the base fluid due to Brownian motion and thermophoresis effects. This results in depletion of thermal boundary layer thickness and hence the temperature overshoots are observed in the vicinity of the stretching sheet for increasing value of $N_b$ and $N_t$. Fig. 6 depicts that temperature reduces when thermal relaxation time for heat flux increases $\gamma$. Physically when $\gamma$ increases the material particles need extra time to transfer heat to its adjacent particles, thus temperature reduces. Fig. 7 shows that a pronounced increase in the temperature and corresponding boundary layer thickness when there is an increase in heat transfer Biot number $\gamma_1$. Fig. 8 shows the impact of Prandtl number $Pr$ on temperature profile. Since $Pr = \frac{\nu}{\alpha}$, thus by increasing Prandtl number $Pr$ thermal conductivity of fluid decreases which reduces temperature profile. Additionally, an increase in Prandtl number reduces thermal boundary layer thickness. So heat rapidly transfers which causes a drop in fluid temperature.
Fig. 5: Effect of $Nt$ on temperature profile.

Fig. 6: Effect of $\gamma_1$ on temperature profile.

Fig. 7: Effect of $\gamma_2$ on temperature profile.

Fig. 8: Effect of $Pr$ on temperature profile.

Figs. 9-11 shows the variation in concentration profile for different values of Schmidt number $Sc$, chemical reaction parameter $k$ and mass transfer Biot number $\gamma_2$. Fig 9 reveals that when Schmidt number increases, it causes decrease in concentration, higher value of Schmidt number lowers the mass diffusivity. Due to this thickness of concentration boundary layer declines. Fig. 10 reveals the influence of chemical reaction parameter $k$ on concentration profile. It shows that concentration decreases with the increase in parameter $k$. This is because the higher the reaction rate, the thicker the concentration boundary layer. It is observed in Fig. 11 that concentration profile rises when mass transfer Biot number $\gamma_2$ increases.

Impacts of parameters $\epsilon$ and $\delta$, Brownian motion parameter $Nb$, Thermophoresis parameter $Nt$ on Nusselt number $-\partial \theta(0)$ are depicted in Figs. 12, 13. Fig. 12 is drawn to examine the variations of $\epsilon$ and $\delta$ on Nusselt number $-\partial \theta(0)$. Nusselt number increases with an increase in $\epsilon$ while Nusselt reduces with an enhancement in $\delta$. It is shown in Fig. 13 that with variations in $Nb$ and $Nt$ Nusselt number increases.

Fig. 14 shows the variation of parameters $\epsilon$ and $\delta$ on Sherwood number $-\varphi'(0)$. It is analyzed that Sherwood number rises with an increase in $\epsilon$ while it reduces with $\delta$.

CONCLUSIONS

In the present study Catteneo-Christov heat flux model is considered for three dimensional flow of Erying Powell fluid. The following observations in the existing study is worth mentioning:

1. Velocity profile in in x and y direction increases with increase in $\epsilon$ while reverse pattern is observed for increasing value of $\delta$.

2. Temperature rises with enhancement in $Nb$, $Nt$ and $\gamma_1$ while it decreases with increment in thermal relaxation time $\gamma$.

3. Concentration profile decreases with increase in $Sc$ (Schmidt number) and $k$ (chemical reaction parameter).
On the other it enhances with mass transfer Biot number $\gamma_2$.

4- Rate of heat transfer i.e., Nusselt number decreases with $N_t$ and it increases with $\epsilon$.

5- Rate of mass transfer i.e., Sherwood number remains almost constant for different values of $\delta$ while it rises with $\epsilon$.

**Nomenclature**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Temperature exponent</td>
</tr>
<tr>
<td>$B$</td>
<td>Concentration exponent</td>
</tr>
<tr>
<td>$c$</td>
<td>Eyring Powell fluid characteristics</td>
</tr>
<tr>
<td>$c_p$</td>
<td>Specific heat of the fluid</td>
</tr>
<tr>
<td>$C_f$</td>
<td>Ambient fluid concentration</td>
</tr>
</tbody>
</table>
REFERENCES


\begin{align*}
h &= \text{Heat transfer coefficient} \\
h^* &= \text{Concentration transfer coefficient} \\
M &= \text{Hartman number} \\
T_f &= \text{Ambient fluid temperature} \\
Sc &= \text{Schmidt number} \\
g &= \text{Gravitational acceleration constant} \\
k &= \text{Chemical reaction parameter} \\
K &= \text{Thermal conductivity of the fluid} \\
\nu &= \text{Local Nusselt number} \\
\bar{nb} &= \text{Brownian motion parameter} \\
\eta &= \text{Thermophoresis parameter} \\
p &= \text{Pressure} \\
Pr &= \text{Prandtl number} \\
q &= \text{Heat flux} \\
Re_x &= \text{Local Reynolds number} \\
Sh &= \text{Local Sherwood number} \\
T &= \text{Temperature} \\
u, v, w &= \text{Velocity components in x, y and z-direction} \\
\alpha &= \text{Thermal diffusivity} \\
\beta &= \text{Erying Powell fluid characteristics} \\
\beta_c &= \text{Concentration expansion coefficient} \\
\beta_e &= \text{Thermal expansion coefficient} \\
\gamma &= \text{Non-dimensional relaxation time} \\
\gamma_1 &= \text{Heat transfer Biot number} \\
\gamma_2 &= \text{Mass transfer Biot number} \\
\delta &= \text{Erying Powell fluid parameter} \\
\epsilon &= \text{Erying Powell fluid parameter} \\
\omega &= \text{Ratio of stretching velocities} \\
\lambda_1 &= \text{Thermal relaxation time} \\
\lambda &= \text{Thermal buoyancy parameter} \\
\lambda_c &= \text{Solutal buoyancy parameter} \\
\vartheta &= \text{Kinematic viscosity} \\
\rho &= \text{Fluid's density} \\
\sigma &= \text{Electrical conductivity} \\
\theta(\eta) &= \text{Dimensionless temperature} \\
\varphi(\eta) &= \text{Dimensionless concentration} \\
\end{align*}

\textbf{Greek Symbols}

\begin{align*}
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\end{align*}

\textbf{Subscripts}

\begin{align*}
()_w &= \text{Wall conditions} \\
()_{\infty} &= \text{Free stream conditions} \\
\end{align*}


