

Pattern Formation of the FitzHugh-Nagumo Model: Cellular Automata Approach

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ABSTRACT: FitzHugh-Nagumo (FHN) model is a famous Reaction-Diffusion System which first introduced for the conduction of electrical impulses along a nerve fiber. This model is also considered as an abstract model for pattern formation. Here, we have used the Cellular Automata method to simulate the pattern formation of the FHN model. It is shown that the pattern of this model is very similar to those of a kind of a rabbitfish which implies natural patterns could be based on reaction-diffusion systems. We have also considered the effects of different parameters of the FHN model on changing the initial pattern.

KEY WORDS: Reaction-diffusion systems, FitzHugh-Nagumo Model, Pattern formation, Cellular automata.

INTRODUCTION

Reaction-Diffusion Systems (RDS) refer to the class of partial differential equations where the right hand side of a time-dependent equation can be divided into a local (in space) “reaction” part and a nonlocal “diffusion” part. The general form of reaction-diffusion equations in one dimension is [1],

$$U_t = f(U) + DU_{xx} \quad (1)$$

where the subscripts denote partial derivatives of the

vector U . The function, $f(U)$ contains the “reaction” terms and the spatial derivatives denote the “diffusion” terms. These equations are an abstract model for pattern formation but in many cases have direct application to the fields of developmental biology, chemistry, optics, and branches of applied mathematics [1-2]. For biological and chemical systems, for example, reaction-diffusion equations represent a reduced description of a complicated set of reactions. The important aspect of all of the applications

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is that the simple combination of reactions plus transport due to diffusion is sufficient to produce a large variety of interesting patterns [3-5].

Scientists try to solve RDS analytically. But in some cases, RDS include sophisticated differential equations, and analytical methods could not lead to results. So, numerical methods may be used. When pattern formation based on RDS is considered, numerical methods seem to be replaced by simulation methods. However, numerical methods are time consuming and advanced mathematics is required, whereas simulation methods like Monte Carlo are generally simpler. Cellular Automata (CA) is one of the simulation methods that seems to be a good one for description of pattern formation. This method was introduced in the last 1960's by *John Von-Neumann* and is widely used to describe new modeling methods [6-8]. A cellular automaton is a 4-Tuple $\{L, S, N, F\}$, where L is an infinite or finite regular lattice of cells, S is a finite set of states. Each cell $i \in L$ is assigned a state $s \in S$. N is a finite set of neighbors, indicating the position of one cell relative to another cell on the lattice (N can and usually does include the reference cell itself). F is a function

$$F: S^{|N|} \mapsto S, \quad \{s_i\}_{i \in N} \mapsto s \quad (2)$$

which assigns a new state to a cell depending on the state of all its neighbors indicated by N and could be a deterministic or probabilistic function. The evolution of a cellular automaton is defined by applying the function F synchronously to all cells of the lattice L . The key features of CA are the regular lattice of cells, each connected to a finite number of neighbors and having a finite state set. The state changes synchronously for all cells at the same time depending only on the state of all neighbors.

One of the most famous reaction-diffusion models is the *FitzHugh-Nagumo (FHN)* model for the conduction of electrical impulses along a nerve fiber [9-11]. The original heuristic model for the flow of electric current through the surface membrane of a giant squid axon was proposed by *Hodgkin & Huxley* in 1952 [12]. The proposed four component model represents equations for the membrane current density, the sodium activation, the sodium inactivation, and the potassium activation in the nerve axon. *FitzHugh* simplified these equations by proposing the following pair of equations he called the *Bonhoeffer-Van der Pol (BVP)* model,

$$\begin{aligned} J &= \frac{1}{c} \frac{\partial u}{\partial t} - u + u^3 - v \\ c \frac{\partial v}{\partial t} + bv &= a - u \end{aligned} \quad (3)$$

where J is the current density, u represents the combined membrane voltage and sodium activation, and v represents the combined sodium inactivation and potassium activation. The parameters a , b , and c are constants (see [9] for details). Using an analog computer and phase plane projections, *FitzHugh* showed that the dynamics of this model were qualitatively the same as the dynamics of the more complicated four variable *Hodgkin-Huxley* model. *Nagumo et al.* studied the same equations but simulated them in an electric circuit as a distributed line [10]. The equations they used show explicitly the diffusion,

$$\begin{aligned} h \frac{\partial^2 u}{\partial s^2} &= \frac{1}{c} \frac{\partial u}{\partial t} - u + u^3 - v \\ c \frac{\partial v}{\partial t} + bv &= a - u \end{aligned} \quad (4)$$

with the constant h related to the density of the resistance per unit length in the line.

Both *FitzHugh & Nagumo* found that this model allows for the propagation of pulses and pulse trains, where the pulses represent an excited domain or electrical signal. The essential feature of this model is that the evolution of u is a cubic function of u and contains an additive linear term of v . The evolution of v has one term proportional to u and one term proportional to v .

In this paper, we have used CA to simulate the pattern formation of the FHN model as a Reaction-Diffusion System. In the next section, a modified FHN model is introduced which we have used for the simulation. After that, we have described the CA method for simulation of the FHN model. Finally, the results are shown and the effects of different parameters on pattern formation are described.

FHN Model

There are many different ways to write and re-scale the equations for the FHN model. For a discussion of some of these different methods see [13]. The equations we have used here are as follow,

$$\frac{\partial u}{\partial t} = D_u \nabla_u^2 + (a - u)(u - 1)u - v \quad (5)$$

$$\frac{\partial v}{\partial t} = D_v \nabla_v^2 + e(bu - v)$$

with $a < 1, e > 0$, and $b \geq 0$.

The reactive dynamics contains three free parameters, related to the position of the intersection of the u - and v -nullcline (here a), the slope of the v -nullcline (here b), and the relative speed of the evolution of u and v (here e) (The nullclines are the curves in phase space where the reactive parts of \dot{u} and \dot{v} in Eq. (5) are zero) [13]. The homogeneous solutions are oscillatory for $a < 0$. For $a \geq 0$, there can be one or two steady states. We only consider $a > 0$. In this case, the state $u_1=0, v_1=0$ is always stable. For $b < (1-a)^2/4$ there exists another stable state at $u_3 = \left(1 + a + \sqrt{(1-a)^2 - 4b}\right)/2$ and $v_3 = bu_3$, as well as an unstable steady state at $u_2 = \left(1 + a - \sqrt{(1-a)^2 - 4b}\right)/2$ and $v_2 = bu_2$. The stability graph for Eq. (5) is drawn in the case of $v \geq 0$ regarding the fact that u and v are related to concentrations and they should not take any negative value (see Fig. 1). As it is shown in the Fig. 1, there are one more stable (minimum energy with greater negative value), one less stable (minimum energy with smaller negative value), and an unstable phase (maximum value of energy) for this system. That is to say, a system in an unstable phase tends to go toward a more stable phase. We will come back to this tendency of the FHN model during the simulation process.

Cellular Automata Model

For simulation of the FHN model as a RDS, we have used a macroscopic approach which consists of a combination of two different steps, a diffusion step and a reaction part [14]. To clarify the separation into two steps, we first discretize the reaction-diffusion equation as follow (for one-variable system),

$$\frac{\partial x(r, t)}{\partial t} = D \nabla^2 x(r, t) + f(x(r, t)) \quad (6)$$

$$x(r, t + \Delta t) = x(r, t) + \Delta t D \nabla^2 x(r, t) + \Delta t f(x(r, t)) + O(\Delta t^2) \quad (7)$$

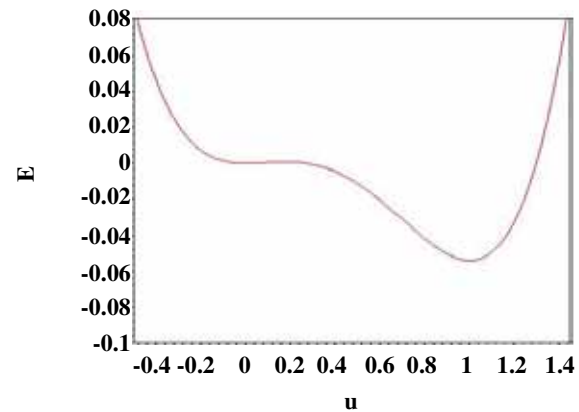


Fig. 1: The graph of energy versus u for Eq. (5) in the case of $v \geq 0$.

On the other hand, to be able to call the simulation method a cellular automaton, and to use a lookup table instead of calculating a nonlinear reaction term, we could discretize the variables. In the diffusion step, each variable $x(r, t)$ is an integer number in the range $[0, M]$ whereas M is called the canal number. In each time step, cell variable value is added to those of its neighbors. After that, the result is distributed equally within the cell and its neighbors. So, the lattice will possess a uniform distribution of particles. As a result, the diffusion operation applied to $x(r, t)$ is approximately [14],

$$\Phi_D(x(r, t)) = c_0 x(r, t) + c_2 \nabla^2 x(r, t) \quad (8)$$

where c_0 and c_2 are the moments of the general set of coefficients with a finite support.

There are also other approaches which use different algorithms for the diffusion step (see for example refs. [14-16]). It should be noted that the effect of the diffusion coefficient for each species must be considered in the diffusion step. That is to say, if a relation between diffusion coefficients of x and y is $D_x/D_y = 2$, it means the diffusion step for x particle takes place two times while y particle does this step just once.

The second step of the cellular automaton simulates the reaction. The operator for the reactive part is defined as [14],

$$\Phi_R(c_0 x(r, t)) = x(r, t) + \Delta t f(x(r, t)) \quad (9)$$

In order to see the accuracy of the sequential application of Φ_D and Φ_R results, it could easily shown that,

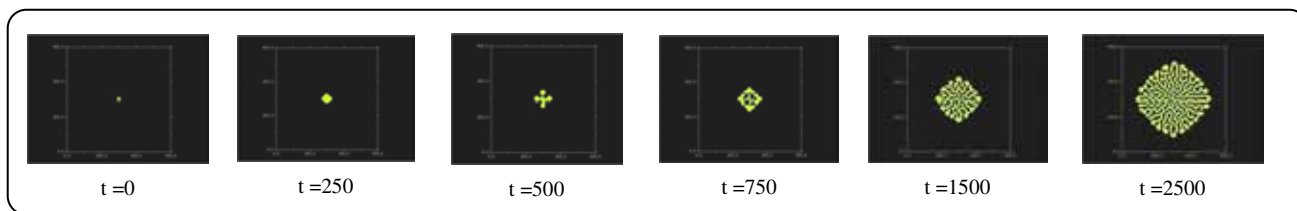


Fig. 2: Formation of the labyrinthine pattern. Simulation of FHN model using macroscopic CA approach. (600×600 square lattice, $M_u=55$, $M_v=256$, $\Delta t=2$, $D_v/D_u=3$, $a=0.18$, $b=0.14$, and $e=0.025$. The simulation starts from a lattice with less stable phase ($u=0$, $v=0$) and a spot of more stable phase ($u=0.75$, $v=0.11$) at the centre of the lattice).

$$\Phi_R(\Phi_D(x(r,t))) = \Phi_R(c_0x(r,t) + c_2\nabla^2x(r,t)) = \quad (10)$$

$$x(r,t) + \frac{c_2}{c_0}\nabla^2x(r,t) + \Delta t f\left(x(r,t) + \frac{c_2}{c_0}\nabla^2x(r,t)\right) =$$

$$x(r,t) + \frac{c_2}{c_0}\nabla^2x(r,t) + \Delta t f(x(r,t)) + O(\Delta t^2)$$

which we can identify with Eq. (7). Using this discretization, it is important to verify that the CA steps respect the discretization, i.e., the outcome of an operation on the integer variables is also an integer. As we mentioned, the operator Φ_D in Eq. (8) acts on a field of integers in the range $[0, M]$ to give a field of integers in the range $[0, c_0M]$. The non-normalized result is also an integer, now in the range $[0, c_0M]$. The normalization, i.e., the multiplication by $1/c_0$ necessary to bring the range back to $[0, M]$, does not preserve the cell values as integers. At this step some mechanism has to be introduced to ensure that the result of the reaction step is an integer in the permitted range. Therefore, it is needed to use another operator Φ_T for truncation. This operator takes the real numbers which result from operator Φ_R and produces an integer. We have used the Probabilistic Minimal Noise Rule (PMNR) for rounding off numbers [14],

$$\Phi_T(x) = \begin{cases} [x] & \text{with probability } (1-p) \\ [x]+1 & \text{with probability } (p) \end{cases} \quad (11)$$

$$(p = x - [x])$$

where $[x]$ denotes a floor function. Therefore, a complete dynamics of a system in a macroscopic CA approach will be given by $\Phi_T\Phi_R\Phi_D$. The detailed algorithm of the CA method for simulation of the FHN model is given in Appendix A.

RESULTS AND DISCUSSION

We have used a 600×600 two-dimensional square lattice with periodic boundary condition. The Moore neighbors are considered for each cell. The canal number is 55 and 256 for u and v species, respectively, where $\Delta t = 2$, $D_v/D_u = 3$, and the values for parameters a , b , and e are 0.18, 0.14, and 0.025, respectively. The initial state consists of a lattice with less stable phase ($u=0$, $v=0$) and a spot of more stable phase ($u=0.75$, $v=0.11$) is located at the centre of the lattice. The simulation is done based on Eqs. (8), (9), and (11). The results are shown in Fig. (2) in which dark and bright colors are related to less and more stable concentrations, respectively. One could see formation of a labyrinthine pattern as a result of movement of a more stable phase among less stable phase. It is interesting to know the labyrinthine pattern is very similar to those of a kind of a rabbitfish (Fig. 3) which means pattern formation in nature could be based on reaction-diffusion systems [3-5].

In the next step, we have considered the effects of different parameters on pattern formation in the FHN model. Let's start with parameter a . This parameter must be in the range 0.17-0.19 in order to form a labyrinthine pattern. For $a < 0.17$, the lattice is covered with more stable phase and a labyrinthine pattern is going to form from the edges of the lattice (Fig. 4a). For $a > 0.19$, just a line of more stable phase is going to grow and there is not any labyrinthine pattern (Fig. 4b). However, the best value for this parameter is 0.18. The effect of parameter b seems to be like a . This parameter must be in the range 0.14-0.15. For $b > 0.15$, growing a line is seen, while in the case $b < 0.14$, growing more stable phase is widely and unusual (Figs. 5a, 5b). The range of the parameter e for occurring labyrinthine pattern is 0.02-0.04 and the value 0.025 is the best. However, there is an interesting case for this parameter. For $e > 0.04$, growing a line of more stable

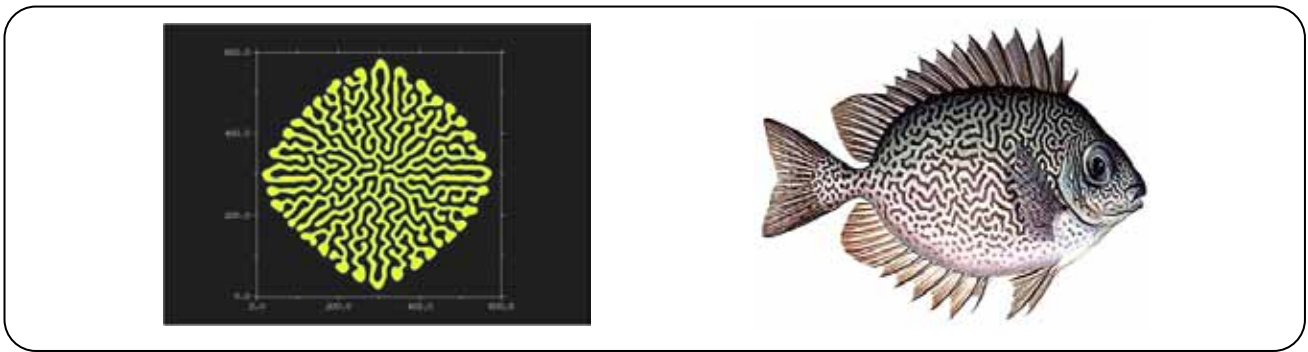


Fig. 3: Comparison of the labyrinthine pattern with the pattern on the skin of a kind of a rabbitfish.

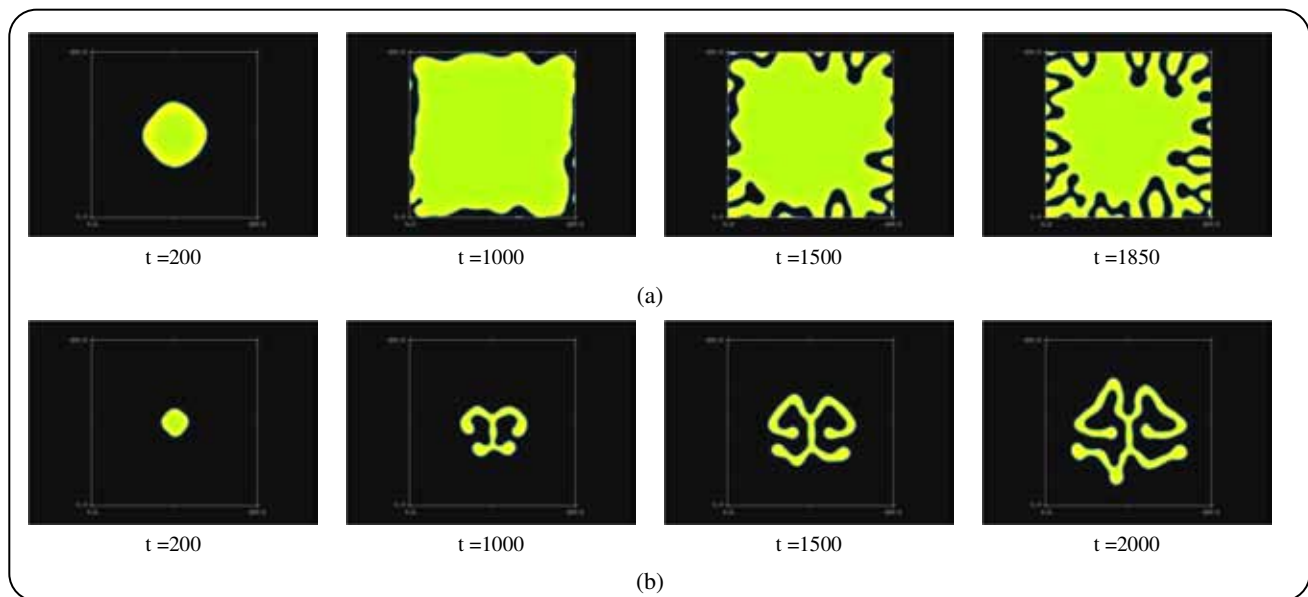


Fig. 4: The effect of parameter a on pattern formation. (a) $a=0.15$ (b) $a=0.2$.

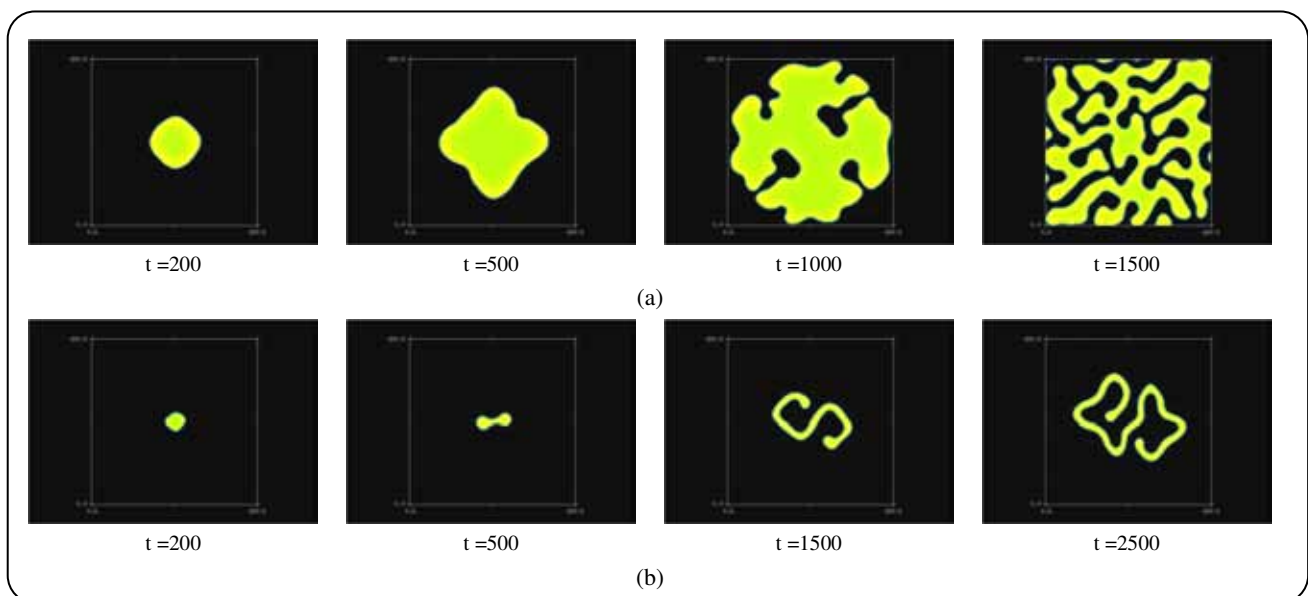


Fig. 5: The effect of parameter b on pattern formation. (a) $b=0.13$ (b) $b=0.16$.

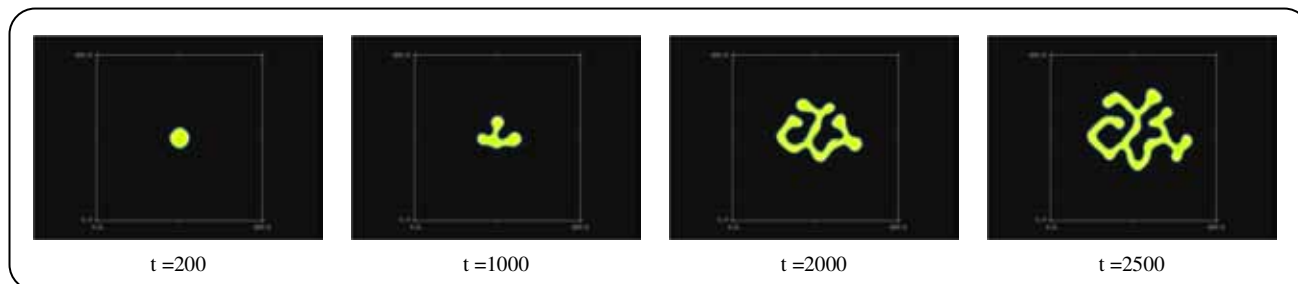


Fig. 6: The effect of parameter e on pattern formation. $e=0.05$.

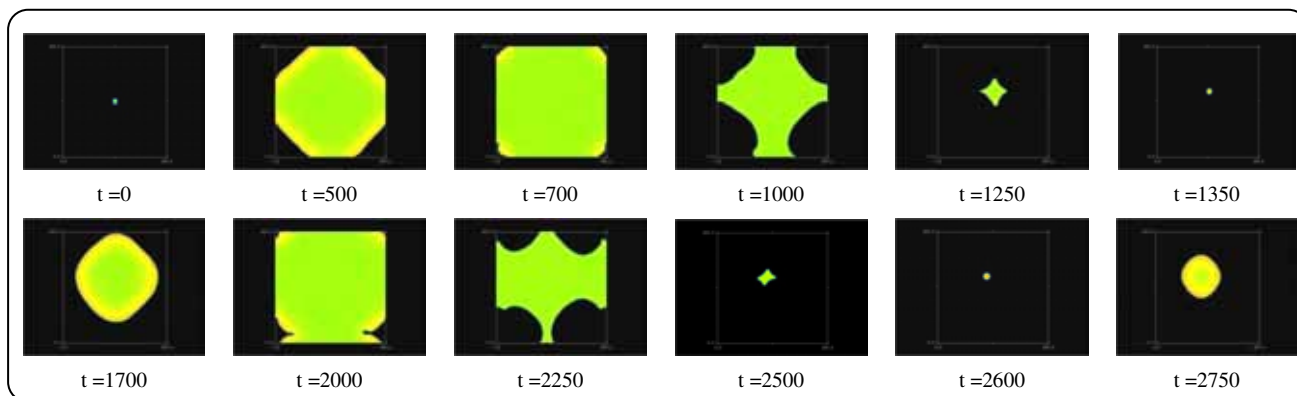


Fig. 7: An oscillation regime in the FHN model. $e=0.01$.

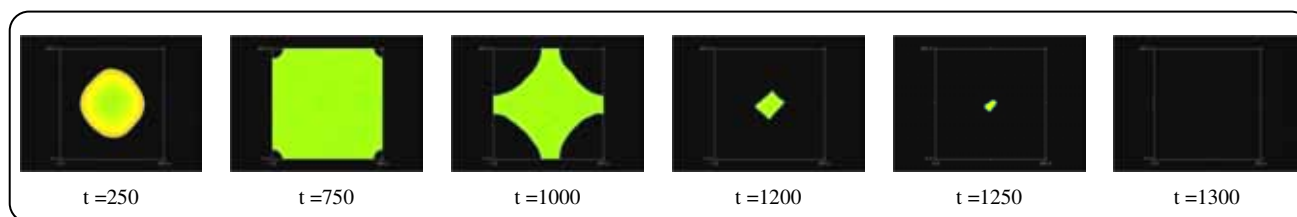


Fig. 8: Sensitivity of the system on parameter e for the oscillation regime. $e=0.009$.

phase is seen (Fig. 6) while in the case $e < 0.02$ there is a different situation. For $e=0.01$, an oscillation will occur. That is to say, the system will be oscillated between two stable phases (Fig. 7). This oscillating regime is very sensitive to the parameter e , whereas it disappears when we use $e=0.009$. In this case, first, the more stable phase will grow. After that, the less stable one will cover the lattice and the system remains unchanged. That is to say, there is just one oscillation behavior in the system (Fig. 8). As we have mentioned before, the value of these parameters and their related patterns could be obtained from a bifurcation analysis [17-18]. We have also considered the effects of other variables on the pattern formation. For the effect of Δt , we have shown that increasing Δt will lead to a fast reaction, as expected. So, one could see the labyrinthine pattern very earlier but

with less symmetry (Fig. 9). Increasing the canal numbers will also lead to more symmetry (Fig. 10). In contrast, decreasing the canal number will cause less symmetric pattern. Moreover, small enough numbers may cause broken pattern (Fig. 11). Broken pattern is also seen in the case $D_v/D_u = 2$ (Fig. 12) which means the diffusion coefficients of species is very important for occurrence the labyrinthine pattern. Finally, a system with the unstable phase at the beginning is simulated. The results are shown in Fig. 13 which shows the occurrence of two stable phases after some time steps, as expected.

Note: the parameters for Figs. 4-13 are as follow (any difference with these values has mentioned in the figure caption):

200×200 square lattice, $M_u=55$, $M_v=256$, $\Delta t=2$, $D_v/D_u = 3$, $a=0.18$, $b=0.14$, and $e=0.025$. The simulation

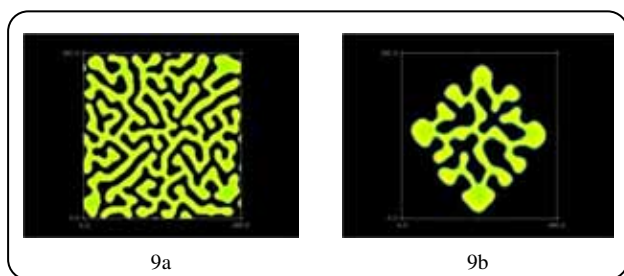


Fig. 9: The effect of Δt on pattern formation in $t=950$. a: $\Delta t=4$, b: $\Delta t=2$.

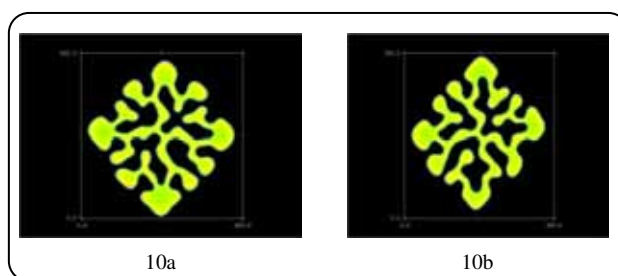


Fig. 10: The effect of canal number on pattern formation in $t=500$. a: $M_v=256$, b: $M_v=155$.

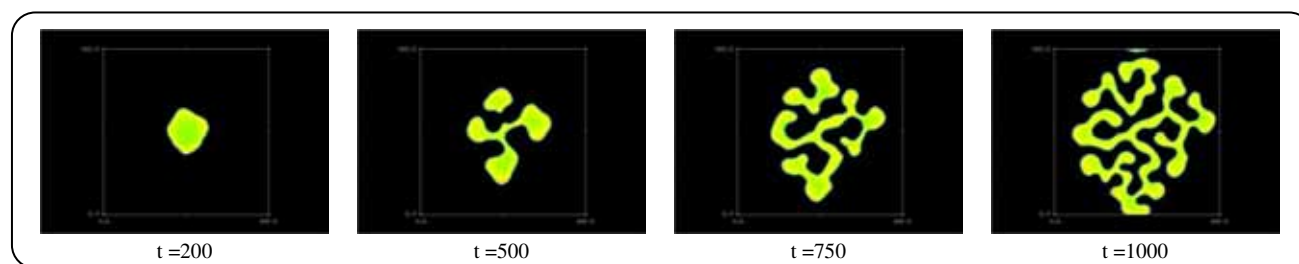


Fig. 11: The effect of canal number on pattern formation which leads to a broken pattern. $M_v=75$.

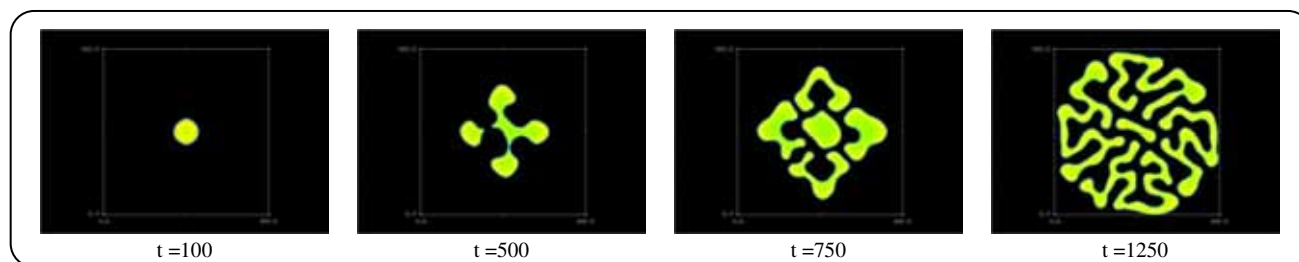


Fig. 12: The effect of diffusion coefficients on pattern formation which leads to a broken pattern. $D_v / D_u = 2$.

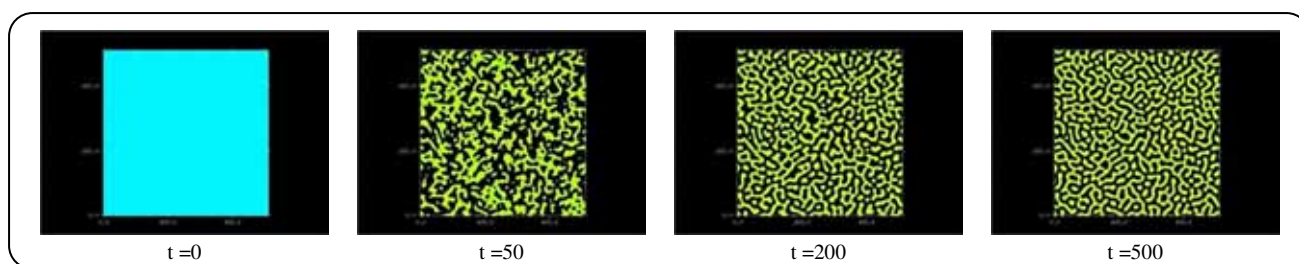


Fig. 13: Simulation with the unstable initial condition. After some time steps, two stable phases are occurred.

starts from a lattice with less stable phase ($u=0, v=0$) and a spot of more stable phase ($u=0.75, v=0.11$) at the centre of the lattice

CONCLUSIONS

We have demonstrated the pattern formation of the FHN model using CA approach. It was shown that the labyrinthine pattern is very similar to the pattern of a kind

of a rabbitfish. This result could imply a relationship between reaction-diffusion systems and natural patterns. We have also described some effects of different parameters on the pattern formation. It was shown an oscillation regime which is very sensitive to the value of the parameter e . There are many cases which could be considered for the FHN model such as movement of one phase among another phase, calculation of the velocity of

the movable phase, considering pattern formation of other FHN models, etc. We are doing some of these cases during these days in order to understand different aspects of the pattern formation for the FHN model.

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Appendix A

Here, the CA algorithm for the FHN model is described;
Definition of different parameters (a, b, e, canal numbers, diffusion coefficients etc)

Definition of the lattice

Definition of the initial state

Beginning of the automata

- Choose all cells one by one
- The diffusion step will be done by knowing the state of the neighbors of a given cell
- The reaction will be done based on Eq. (9) and the obtained results will round off according to Eq. (11)

(The automata will be done till the end of the time steps)

End of the automata

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